Competing dual gradient system in chemotaxis

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December 29, 2014

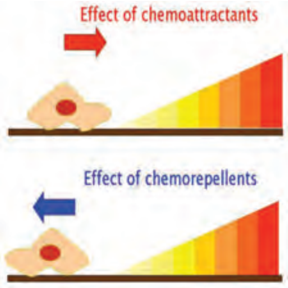
What is chemotaxis?

Chemotaxis: directed movement of cells/organisms in response to the chemical concentration gradient.

Application: Wound healing, cancer growth (or metastasis), embryo development, bacterial movement, predator-prey system, etc.

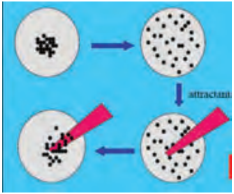
1. attractive chemotaxis: if movement is up the chemical concentration gradient;

2. repulsive chemotaxis: if movement is down the chemical concentration gradient.

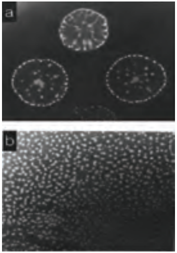


Typical patterns of chemotaxis

Attractive chemotaxis: aggregation and wave propagation



Experiments for E. Coli (by Adler 1966 and Berg-Budrene 1995):



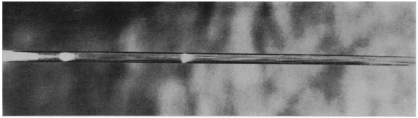


Fig. 1. Photograph showing bands of *E. coli* in a capillary tube. In all the experiments reported here, capillary tubes (*18*) were filled with a liquid medium (*19*), inoculated at one end with 2×10^5 to 2×10^8 bacteria (*20*), and then closed at the ends with plugs of agar and clay, all according to a procedure described in full elsewhere (*8*). The tubes were incubated horizontally at 37°C. The origin, which is turbid because of the bacteria that have not moved out, is visible at the left, then the second band of bacteria, then the first band. Plugs at ends are not shown. The concentration of galactose was 2.5×10^{-4} mole per liter.

Repulsive chemotaxis: uniformization/homogenization

Chemotaxis model (By Keller and Segel in 1971)

Let u denote cell (particle) density and v chemical concentration. If the cell kinetics (growth and death) is ignored, then

$$u_t + \nabla \cdot J = 0$$

where J denotes the cell flux which is made up of two parts

$$J = J_{\text{diffusion}} + J_{\text{chemotaxis}}.$$

Here diffusive flux

$$J_{\text{diffusion}} = -D(u, v)\nabla u$$

was due to Fick's law and describesthe random dispersion of cells, and the chemotactic flux

$$J_{\text{chemotaxis}} = \chi u \Phi(v) \nabla v =: \chi u \nabla \phi(v)$$

contributes a directed movement due to the presence of chemical concentration gradient.

The model governing the chemical dynamics is a reaction-diffusion equation

$$v_t = \varepsilon \Delta v + g(u, v).$$

Chemotaxis model

The coupling of above equations gives the following chemotaxis system: Keller-Segel model

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \nabla \phi(v)), \\ v_t = \varepsilon \Delta v + g(u, v). \end{cases}$$

u - cell density; v - chemical concentration;

χ - chemotactic coefficient $\begin{cases} \chi > 0 - \text{attractive} \\ \chi < 0 - \text{repulsive} \end{cases}$

$\phi(v)$ - chemotactic sensitivity (potential) function;

ε - chemical diffusion coefficients;

Frequently used forms of $\phi(v)$:

1. Linear law: $\phi(v) = v$ (aggregation); $\Rightarrow$ "Classical (minimal) Keller-Segel model" if $g(u, v) = \alpha u - \beta v$;
2. Logarithmic (Weber-Fechner) law: $\phi(v) = \log v$ (wave propagation);
3. Receptor law: $\phi(v) = \frac{v}{1+v}$;

More forms, see "Tindall-Maini-Porter-Armitage, Bull. Math. Biol., 70:1570-1607, 2008".

Minimal parabolic-elliptic chemotaxis model

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \nabla v), & x \in \Omega \\ \tau v_t = \Delta v + \alpha u - \beta v, & x \in \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega \end{cases}$$

where ν denotes the outward normal vector of $\partial\Omega$.

- Jäger and Luckhaus 1992 [1]: $N = 2$ and β is of order τ and α is of order 1, for the limiting case $\tau \to 0$,

$$\begin{cases} u_t & = \nabla \cdot (D\nabla u - \chi u \nabla v^*), \\ 0 & = \Delta v^* + \alpha(u - \overline{u_0}) \end{cases}$$

  where $v^* = v - \bar{v}(t)$, $\bar{f} = \frac{1}{\Omega} \int_\Omega f(x)dx$.
  In radially symmetric case, initial values were constructed such that blow-up of u occurs in finite time.

[1] W. Jäger and S. Luckhaus, On explosion of solutions to a system of partial differential equations modeling chemotaxis, Trans. Amer. Math. Soc., 329: 819-824, 1992.

Minimal parabolic-elliptic chemotaxis model

- α and β are of the same order 1: T. Nagai (1995) [2] $\Longrightarrow$ Parabolic-elliptic KS model

$$\begin{cases} u_t = \nabla\cdot(D\nabla u - \chi u\nabla v), & x\in\Omega \\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega \end{cases}$$

where $\Omega = B_0(R)$ is the open ball of radius R centered at the origin.

1. When $N=1$ and $N=2$ and $\int_\Omega u_0(x)dx < \frac{8\pi D}{\alpha\chi}$, then;
2. $N=2$, there is a critical number (mass) $\theta = \frac{8\pi D}{\alpha\chi}$ such that solution exists globally with a uniform-in-time bound if $\int_\Omega u_0(x)dx < \theta$, and solution blows up at the origin $x=0$ if $\int_\Omega u_0(x)dx > \theta$;
3. When $N\geq 3$, the solution blows up in finite time at the origin $x=0$ if $0 < M_N(0) = \int_{\omega_N} u_0(x)|x|^k dx < c(\bar{u}_0)$ where ω_N is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ and $\bar{u}_0 = M_0(0) = \int_{\omega_N} u_0(x)dx$;
4. The above results are valid for the Jäger-Luckhaus's model.

[2] T. Nagai, Blow-ip of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., 2: 581-601, 1995.

Minimal parabolic-elliptic chemotaxis model

In a general smooth domain Ω (non-radial solutions)

$$\begin{cases} u_t = \nabla\cdot(D\nabla u - \chi u\nabla v), & x\in\Omega \\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega \end{cases}$$

- T. Nagai (1997) [3]: Let $\int_\Omega u_0(x)|x-q|dx$ be sufficiently small (meaning the mass is concentrated at $x=q$ since otherwise the smallness can not be ensured). Then
  1. If $q\in\Omega$, solution blows up in finite time if $\int_\Omega u_0(x)dx > \frac{8\pi D}{\alpha\chi}$;
  2. If $q\in\partial\Omega$, solution blows up in finite time if $\int_\Omega u_0(x)dx > \frac{4\pi D}{\alpha\chi}$;
- P. Biler (1998) [4]: solution exist globally if

$$\int_\Omega u_0(x)dx < \begin{cases} \frac{8\pi D}{\alpha\chi}, & \text{if } u_0 \text{ is radially symmetric} \\ \frac{4\pi D}{\alpha\chi}, & \text{otherwise} \end{cases}$$

[3] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic system modeling chemotaxis in two-dimensional domains, J. Inequal & Appl., 6: 37-55, 2001.

[4] P. Biler, Local and global solvability of some parabolic systems modeling chemotaxis, Adv. Math. Sci. Appl., 8: 715-743, 1998.

Classical (minimal) parabolic-parabolic KS model

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \nabla v), & x \in \Omega \subset \mathbb{R}^N, t > 0 \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega \subset \mathbb{R}^N, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

Attractive case $(\chi > 0)$:

- $N = 1$, uniform boundedness (Osaki-Yagi 2001, Funkcial. Ekvac.)
- $N = 2$ $(M = \int_\Omega u_0(x)dx) \Longrightarrow$
  1. Nagai, Senba and Yoshida [5]: Global solution exists with uniform-in-time bound if $M < \frac{4\pi\varepsilon}{\alpha\chi}$, or $M < \frac{8\pi\varepsilon}{\alpha\chi}$ if Ω is a disk.
  2. Herrero and Velázquez (1996) [6]: construction of blow-up solution and show that $\frac{8\pi\varepsilon}{\alpha\chi}$ is the critical mass if Ω is a disk in $\mathbb{R}^2$.
  3. Horstmann and Wang [7]: For every $M > \frac{4\pi\varepsilon}{\alpha\chi}$ and $M \neq \frac{4k\pi\varepsilon}{\alpha\chi}$, $k \in \mathbb{N}^+$, solution may blow up either in finite or infinite time.

[5] T. Nagai, T. Senba and K. Yoshida, Applications of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funcialaj Ekvacioj, 40: 411-433, 1997

[6] M.A. Herrero and J.J.L. Velázquez. Chemotactic collapse for the Keller-Segel model. J. Math. Biol., 35:177-194, 1996.

[7] D. Horstmann and G. Wang. Blow-up in a chemotaxis model without symmetric assumptions. European J. Appl. Math., 12:159-177, 2001.

Classical parabolic-parabolic KS model

Open question for $N = 2$: whether the blowup time is finite or infinite?

- $N \geq 3$:
  1. M. Winkler (2010)[8]: $N \geq 3$ and Ω is a ball, the solution may blow up in finite or infinite time for any $M = \int_\Omega u_0 dx > 0$.
  2. M. Winkler (2013)[9]: $N \geq 3$ Ω is a ball, radial solution blows up in finite time for any $M = \int_\Omega u_0(x) dx > 0$;

Remark: All above works essentially used the time-monotone Lyapunov (energy) functional.

Comprehensive review of all detailed results, see "Horstmann 2003, Jahresbericht Der DMV ".

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[8] M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. J. Diff. Eqns., 248: 2889-2905, 2010.

[9] M. Winkler. Finite-time blow-up in the higher-dimensional parabolic parabolic Keller-Segel system, J. Math. Pures. Appl., 100: 748-767, 2013

Classical parabolic-parabolic KS model

Repulsive Keller-Segel model:

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \nabla v), & x \in \Omega \subset \mathbb{R}^N, t > 0 \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega \subset \mathbb{R}^N, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

Repulsive case ($\chi < 0$): Cieślak, Laurençot and Morales-Rodrigo [10]

- $N = 1, 2$: Global solutions exist and converge to the unique constant stationary solution exponentially as $t \to \infty$;
- $N = 3, 4$: Global weak solutions exist.

Note: The time-monotone Lyapunov functional was essentially used.

[10] T. Cieślak, P. Laurençot and C. Morales-Rodrigo, Global Existence and Convergence to Steady-States in a Chemo-repulsion System, Banach Center Publications, 81 (Polish Acad. Sci.):105-117, 2008

Attraction-Repulsion Keller-Segel (ARKS) model

"Attraction-Repulsion Keller-Segel model":

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u \nabla w), & x\in\Omega,\ t>0, \\ v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0, \\ w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0. \end{cases}$$

Why is this model of interest and importance?

- Luca *et al*, Bull. Math. Biol. 2003: aggregation of microglia and formation of local accumulation of chemicals observed in Alzheimer's disease (u-density of Microglia, v-concentration of Interleukin-1β, w-concentration of Tumor necrosis factor-α);
- Painter-Hillen, Canadian Appl. Math. Q 2003: quorum sensing effects in chemotaxis;

Mathematical challenges: no obvious Lyapunov functional exist and most (if not all) existing approaches are not applicable directly.

First results: Tao-Wang, Math. Models Methods. Appl. Sci. (M^3AS), 23:1-36, 2013: $\beta = \delta$ and $\beta \neq \delta$.

ARKS model

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega,\ t > 0, \\
v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\
w_t = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
(u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega.
\end{cases}
$$

It was found by Tao-Wang that the solution behavior of the ARKS model was essentially determined by the competition of attraction and repulsion which is characterized by the sign of $\chi\alpha - \xi\gamma$. The number

$$\theta = \chi\alpha - \xi\gamma$$

is defined as the competition index. The biological interpretation of the sign of θ is as follows:

- $\theta > 0 \Leftrightarrow$ attraction dominates;
- $\theta = 0 \Leftrightarrow$ repulsion balances/cancels attraction;
- $\theta < 0 \Leftrightarrow$ repulsion dominates;

Main ideas for case of $\beta = \delta$ (same death rates)

1. If $\theta \neq 0$, then set $s = \theta(\xi w - \chi v)$. The ARKS model becomes

$$\begin{cases} u_t = \Delta u - \theta \nabla \cdot (u \nabla s), \\ s_t = \Delta s - \delta s + u, \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w. \end{cases}$$

Observation: The first two equations constitute an exact classical KS model, and hence existing approaches (i.e. Lyapunov functional approach) on the classical KS model can be applied.

2. If $\theta = 0$, then set $s = \xi w - \chi v$ and the model becomes

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla s), \\ s_t = \Delta s - \delta s, \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w. \end{cases}$$

Results for case of $\beta = \delta$ / Tao-Wang (2013)

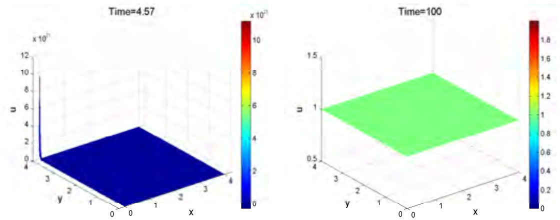
- (Stationary solutions (S.S)) Let $N \geq 1$. Then

  (1) If $\theta > 0$ (attraction dominates), there is a non-constant S.S.

  (2) If $\theta < 0$ (repulsion dominates) or $\theta = 0$ (repulsion balances attraction), there is a unique constant S.S $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\beta}\bar{u}_0)$.

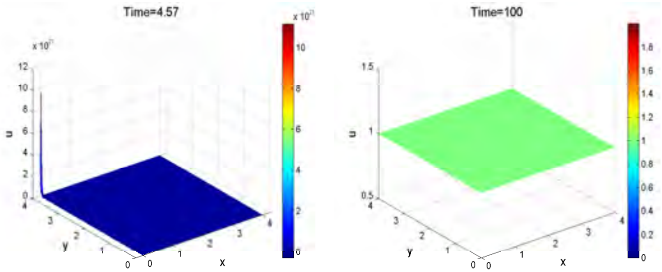
- (Large-time behavior) Let $N = 2$ and $0 \leq u_0, v_0, w_0 \in W^{1,\infty}(\Omega)$. If $\theta \leq 0$ (repulsion dominates/cancels attraction), there is a unique non-negative classical solution which converges to $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\beta}\bar{u}_0)$ as $t \to \infty$.

- (Blow-up) Let $N = 2$ and $\theta > 0$ (attraction dominates). If $\int_\Omega u_0 > \frac{8\pi}{\theta}$, the solution blows up in finite/infinite time.



$\theta > 0$ $\qquad\qquad$ $\theta \leq 0$

Important Question



$\theta > 0$ $\qquad\qquad$ $\theta \leq 0$

Conclusion: there is no pattern formation in 2-D when $\beta = \delta$.

Question: Is there pattern formation in 2-D for $\beta \neq \delta$.

Pattern formation

Pattern formation: a constant equilibrium loses stability under spatially inhomogeneous perturbations when a parameter changes, and some stable non-constant spatially non-homogeneous solutions arise.

Method: Linear stability analysis (identifying the instability regime of parameters)+Global (or Hopf) bifurcation theory (rigorous analysis).

Theorem [Liu, Shi and Wang, DCDS-B, 2013]. Let $(\bar{u}, \bar{v}, \bar{w})$ be a positive constant equilibrium point and define

$$A^* =: A^*(\beta, \delta) = \frac{(p^* + \delta)^2(2p^* + \beta)}{(\beta - \delta)p^*}$$

where p^* is the unique positive root of the equation

$$4p^3 + (4\delta + \beta)p^2 = \delta^2\beta.$$

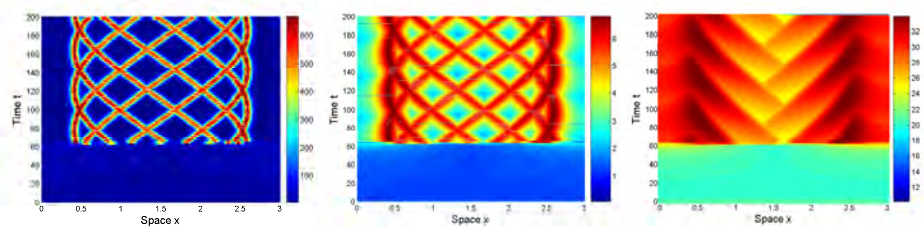
Then we have for $N = 1$:

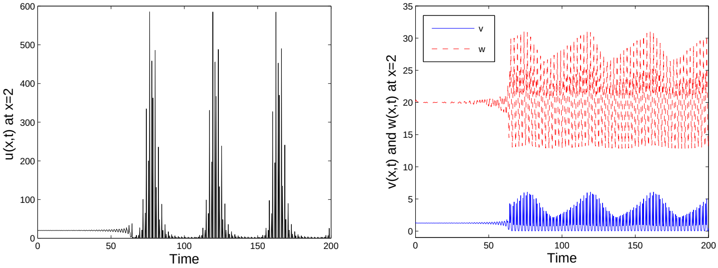
(a) $\beta > \delta$ and $\xi\gamma\bar{u} > A^*$; $\Rightarrow$ non-constant steady state

(b) $\delta \geq \beta$ or $\beta > \delta$ and $\xi\gamma\bar{u} < A^*$; $\Rightarrow$ time-periodic pattern

Periodic rippling patterns for case (a)

$$\alpha = 1, \beta = 16, \gamma = 1, \delta = 1, \xi = 1, \bar{u} = 20, \chi = 3 \Longrightarrow$$





Open Question: Pattern formation in 2-D?

Case of $\beta \neq \delta$ (lack of Lyapunov functional)

Case 1: Parabolic-Elliptic-Elliptic (P-E-E) model:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega,\ t > 0, \\ 0 = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ 0 = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

- Tao-W. (2013): Assume that $0 \leq u_0(x) \in W^{1,\infty}(\Omega)$ and $\theta = \chi\alpha - \xi\gamma \leq 0$ (repulsion dominates or cancels attraction). Then for any $n \geq 2$ and any $\beta \geq 0, \delta \geq 0$, there exists a unique classical non-negative solution (u, v, w) in $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$.

  Main idea of proof: L^p-estimates + parabolic and elliptic regularity.

- Espejo and Suzuki (2014, AML): If $\theta = \chi\alpha - \xi\gamma > 0$ (attraction dominates), the number $\dfrac{8\pi}{\theta}$ is the critical mass;

Case of $\beta \neq \delta$

Case 2: Parabolic-Parabolic-Parabolic (P-P-P) ARKS model:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ w_t = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega. \end{cases}$$

- Tao-Wang (2013): Let $N = 2$ and $\theta = \chi\alpha - \xi\gamma < 0$ (repulsion dominates). Assume that $0 \leq u_0, v_0, w_0 \in W^{1,\infty}(\Omega)$ with $\int_\Omega u_0 \leq \frac{2\beta^2(\xi\gamma - \chi\alpha)}{C_{GN}\chi^2\alpha^2(\beta-\delta)^2}$. Then the P-P-P model admits a unique global classical solution, where C_{GN} is determined from: $\|f\|^4_{L^4(\Omega)} \leq C_{GN}(\|\nabla f\|^2_{L^2(\Omega)}\|f\|^2_{L^2(\Omega)} + \|f\|^4_{L^2(\Omega)})$.
  Main idea: Entropy inequality+Moser iteration+regularity theory.
- Liu-Tao (2014) and Jin (2014): Global classical solutions for any $\int_\Omega u_0 dx > 0$.

Open: Dynamics for $\theta = \chi\alpha - \xi\gamma > 0$ (attraction dominates).

Case of $\beta \neq \delta$

Case 3: Parabolic-Parabolic-Elliptic (P-P-E) ARKS model:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ (u,v)(x,0) = (u_0,v_0)(x), & x\in\Omega. \end{cases}$$

- Jin-Wang (2014): Assume that $0 \le (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ and $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Then if $\theta \le 0$ (repulsion dominates or balances attraction), there P-P-E ARKS model has a unique classical solution $(u, v, w) \in C(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))$ such that

$$\|u(\cdot,t)\|_{L^\infty} \le C$$

where C is a constant independent of t.

Case of $\beta \neq \delta$

Case 3: Parabolic-Parabolic-Elliptic (P-P-E) ARKS model:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0, \\ v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0, \\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ (u,v)(x,0) = (u_0,v_0)(x), & x\in\Omega. \end{cases}$$

- Jin-Wang (2014): Assume that $0 \le (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ and $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Let $M = \int_\Omega u_0(x)dx$. If $\theta > 0$ (attraction dominates), then the following alternatives hold:

  (i) If $M < \frac{4\pi}{\theta}$ (subcritical mass) then the P-P-E ARKS model admits a unique classical solution $(u,v,w) \in C(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))$ such that $\|u(\cdot,t)\|_{L^\infty} \le C$ for a constant C independent of t.

  (ii) If $M > \frac{4\pi}{\theta}$ (supercritical mass) and $M \notin \{\frac{4\pi m}{\theta} : m \in \mathbb{N}^+\}$ where $\mathbb{N}^+$ denotes the set of positive integers, then there exist initial data such that the solution of the P-P-E ARKS model blows up in finite or infinite time.

Sketch of ideas: boundedness for subcritical mass

Lemma (Key Lemma)

If there is a constant $C_1 > 0$ such that the solution of P-P-E ARKS model satisfies

$$\|u \ln u\|_{L^1} + \int_0^t \|v_t(\tau)\|_{L^2}^2 d\tau \le C_1, \tag{1}$$

then there exists a constant $C_2 > 0$ such that

$$\|u\|_{L^2} \le C_2.$$

Procedure of proving boundedness:

$$\|u\|_{L^2} \xrightarrow[\text{Estimates}]{\text{Energy}} \|u\|_{L^3} \xrightarrow[\text{Theorem}]{\text{Regularity}} \|(\nabla v, \nabla w)\|_{L^\infty} \xrightarrow[\text{Iteration}]{\text{Moser}} \|u\|_{L^\infty}$$

Key: Prove (1).
How: Lyapunov functional:

$$F(u,v,w) = \int_\Omega u \ln u dx + \frac{\chi}{2\alpha}\int_\Omega (\beta v^2 + |\nabla v|^2)dx + \frac{\xi}{2\gamma}\int_\Omega (\delta w^2 + |\nabla w|^2)dx - \chi \int_\Omega uvdx.$$

Sketch of ideas: blowup for supercritical mass

Lemma 1 (Energy decay)

Suppose that (u, v, w) is a global and bounded solution of the P-P-E ARKS model. Then there exists a sequence of times $t_k \to \infty$ and nonnegative function $(u_\infty, v_\infty, w_\infty) \in [C^2(\bar{\Omega})]^3$ such that $(u(\cdot, t_k), v(\cdot, t_k), w(\cdot, t_k)) \to (u_\infty, v_\infty, w_\infty)$ in $[C^2(\bar{\Omega})]^3$. Furthermore, $(u_\infty, v_\infty, w_\infty)$ is a steady state of the P-P-E ARKS model, such that

$$F(u_\infty, v_\infty, w_\infty) \leq F(u_0, v_0, w_0).$$

Use the well-known function (Chen and Li)

$$\psi_\varepsilon(x) = \ln\left(\frac{8\pi\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2}\right), \ \varepsilon > 0, x_0 \in \mathbb{R}^2.$$

which is the solution of

$$\begin{cases} -\Delta\psi(x) = e^{\psi(x)}, & x \in \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{\psi(x)} dx < \infty. \end{cases}$$

We construct the sequence:

Sketch of ideas: blowup for supercritical mass

$$\begin{cases} v_\varepsilon(x) &= \frac{\alpha}{\theta}\left(\psi_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega \psi_\varepsilon(x)dx\right) \\ u_\varepsilon(x) &= \frac{Me^{\tilde{\theta}v_\varepsilon(x)}}{\int_\Omega e^{\tilde{\theta}v_\varepsilon(x)}dx}, \quad \tilde{\theta} = \frac{\alpha}{\theta} \\ w_\varepsilon(x) &= \frac{\gamma}{\alpha}v_\varepsilon(x) \end{cases}$$

and then show

Lemma 2 (large negative steady-state energy)

Assume $M > \frac{4\pi}{\theta}$. If $x_0 \in \partial\Omega$, then as $\varepsilon \to 0$, it follows that

$$F(u_\varepsilon, v_\varepsilon, w_\varepsilon) \to -\infty \ \text{ and } \ \int_\Omega |\nabla v_\varepsilon|^2 dx = \frac{\alpha}{\gamma}\int_\Omega |\nabla w_\varepsilon|^2 dx \to \infty.$$

Moreover

Lemma 3 (Lower bound for steady-state energy):

Suppose $M \neq \frac{4\pi m}{\theta}$ for all $m \in \mathbb{N}^+$. Let $(\tilde{u}, \tilde{v}, \tilde{w})$ be a steady state of ARKS model. Then there exists a constant $K > 0$ such that

$$F(\tilde{u}, \tilde{v}, \tilde{w}) \geq -K.$$

Sketch of ideas: blowup for supercritical mass

Theorem

Assume $M > \frac{4\pi}{\theta}$ and $M \notin \{\frac{4\pi m}{\theta} : m \in \mathbb{N}^+\}$. Then there exists initial data (u_0, v_0) such that the corresponding solution of the P-P-E ARKS model blows up (in finite or infinite time).

Proof. By Lemma 2, we can find ε_0 such that

$$F(u_{\varepsilon_0}, v_{\varepsilon_0}, w_{\varepsilon_0}) < -K\,.$$

Define $(u_0, v_0, w_0) = (u_{\varepsilon_0}, v_{\varepsilon_0}, v_{\varepsilon_0})$. If (u, v, w) is globally bounded, then by Lemma 1, we have

$$F(u_\infty, v_\infty, w_\infty) \leq F(u_0, v_0, w_0) < -K$$

which contradicts Lemma 2.

Summary and future works

Conclusions:

- Time-periodic pattern can be found in the ARKS model, which was impossible for the classical KS models with one chemical due to the existence of time-monotone of Lyapunov functional;
- Death rates of two chemicals are crucial for the pattern formation.

Other works:

- R. Shi and W. Wang, well-posedness of ARKS model on unbounded domain;
- X. Li and Z. Xiang, ARKS model with cell kinetics;
- Wacher and Kaja (2012), Numerical computation of ARKS model.

Interesting opening questions:

- The solution behavior for the P-P-P system when $\xi\gamma - \chi\alpha < 0$;
- Amendment of the ARKS model such that aggregation or wave propagation patters are possible;

Ongoing works

Modified attraction-repulsion chemotaxis model such that:

- Attraction and repulsion has a dynamical interaction, and aggregation is generated regardless of the sign of θ (domination);

We consider the following attraction-repulsion chemotaxis model:

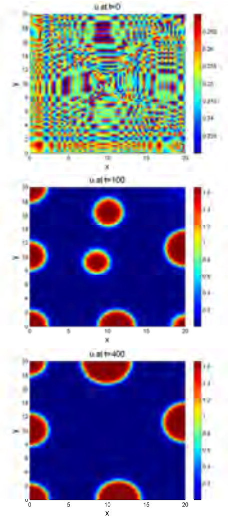
$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u^m \nabla w), & x \in \Omega,\ t > 0, \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u, \tau_1 v, \tau_2 w)(x, 0) = (u_0, \tau_1 v_0, \tau_2 w_0)(x), & x \in \Omega. \end{cases}$$

Theorem

Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 1)$ and $0 \leq u_0 \in W^{1,\infty}(\Omega)$. Then above system with $\tau_1 = \tau_2 = 0$ has a classical solution $(u, v, w) \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ for each $m > 1$ and $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$, and there is a constant c independent of t such that $\|u(\cdot, t)\|_{L^\infty} < c$.

Numerical pattern formation $\theta = \chi\alpha - \xi\gamma$

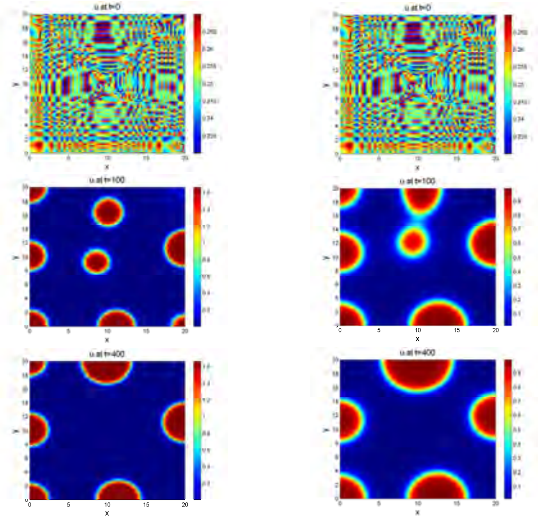
$\theta > 0$



Note. The modified model does show the pattern formation no matter how attraction and repulsion interact.

Numerical pattern formation $\theta = \chi\alpha - \xi\gamma$

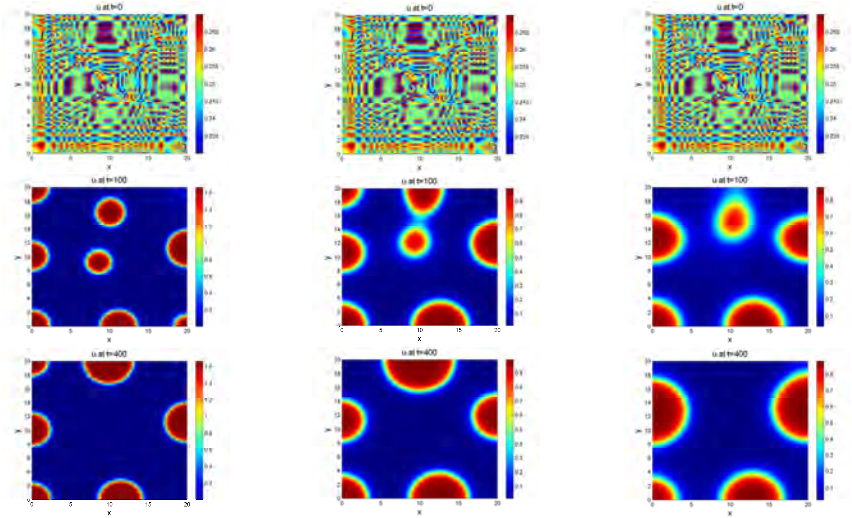
$\theta > 0$ $\qquad\qquad$ $\theta = 0$



Note. The modified model does show the pattern formation no matter how attraction and repulsion interact.

Numerical pattern formation $\theta = \chi\alpha - \xi\gamma$

$\theta > 0$ | $\theta = 0$ | $\theta < 0$



Note. The modified model does show the pattern formation no matter how attraction and repulsion interact.

Acknowledgement

- Thank you for your attention:

- Invitation by Prof. Tai-Chia Lin
- Hong Kong RGC grant.